Chapter 3

Optimization without constraints

When facing problems of the real life, investors, and more generally economic agents, try to do their best. This means that they try to take the best decision, taking into account the information they have.

In most cases, the real world is too complex to be entirely embedded in the formulation of the optimization problem. Models of decision making use simplified representations of the real world. In these simplified frameworks, taking a decision often means maximizing or minimizing a function depending on several variables.

In microeconomics, all the theory is based on the assumption that agents maximize their expected utility. The utility functions are assumed concave and, of course, non-linear, because of the decreasing marginal utility.

In Markowitz portfolio theory[1], investors minimize the risk (measured by the variance of returns) of their portfolio, being given a threshold of expected return they want to reach. Equivalently, the problem can be solved by maximizing the expected return, being given a level of risk the investor accepts to bear.

[1] Markowitz, H.(1952), Portfolio Selection, *Journal of Finance, 7(1), 77-91.*

Download free eBooks at bookboon.com

In corporate finance, firms try to maximize their profits but have to take into account the inverse relationship between the prices of the products they sell and the demand for these products. The firms also try to minimize their costs which are decomposed between fixed and variable costs. In general, decisions that decrease fixed costs have a tendency to increase variable costs. Solving this kind of problem is a matter of optimization.

All these examples show that economic life is paved with the resolution of optimization problems. These problems may include constraints on the possible values of decision variables.

This chapter is devoted to the methods adapted to the resolution of non-linear optimization problems. We assume that no constraint on the decision variables makes the problem more complex to solve. The following chapter will be devoted to these constrained optimization problems.

For the sake of simplicity, we start by single-variable optimization. In principle, the reader already knows these preliminary results. They are intuitive if derivatives of functions had been well understood.

Optimizing functions of several variables is a little bit more difficult because optimality conditions are related to partial derivatives and to the Hessian matrix. Here too, these optimality conditions are natural if partial derivatives and Hessian matrix are understood.

3.1 Preliminaries

3.1.1 The domain of optimization

In general, the functions f to be optimized are defined on a domain $D \subset \mathbb{R}^n$, and take their values in $\mathbb{R}$. An optimization problem can then be written in one of the two following ways:

$$\boxed{\max_{x\in D} f(x) \text{ or } \min_{x\in D} f(x)}$$

Download free eBooks at bookboon.com

The first formulation means that we look for $x^* \in D$ such that $f(x^*)$ is the maximum value taken by f on the domain D; in the second formulation we look for $x^* \in D$ such that $f(x^*)$ is the minimum value taken by f on D.

As already mentioned, D can be equal to $\mathbb{R}^n$, but in most cases D is a subset of $\mathbb{R}^n$, either because f is not defined on all $\mathbb{R}^n$ or because of the characteristics of the problem. For example, when minimizing the risk of a portfolio, assuming that shortsales are forbidden imply restrictions on the domain D.

Download free eBooks at bookboon.com
Click on the ad to read more

Let the function f be defined by:

$$f(x) = \sqrt{x_1 x_2}$$

D is written as follows:

$$D = \left\{ x \in \mathbb{R}^2 \ / \ x_1 x_2 \geq 0 \right\}$$

because square roots are defined only for positive numbers.

Optimization criteria also depend on the shape of the domain D. More precisely, the fact that D is open or closed has an influence on the existence of a solution to the problem at hand.

To illustrate this remark, consider the following problem:

$$\min_{x \in D} f(x) = x^2 - 4 \quad (3.1)$$

$$D = [-3; +2] \quad (3.2)$$

This function has a minimum value equal to -4 for $x = 0$, that is $f(0) = -4 \leq f(x)$ for any $x \in D$. Figure 3.1 shows the graph of the function; you can observe that the first derivative of f at $x = 0$ is equal to 0. In fact, f' is given by:

$$f'(x) = 2x \quad (3.3)$$

Moreover, the sign of the derivative changes at $x^* = 0$ but f" is always positive (f"$(x) = 2$). The function f is then convex and "easier" to minimize.

Suppose now that you are looking for a maximum. Figure 3.1 shows that the maximum is reached for $x^{**} = -3$ with $f(x^{**}) = 5$. However, the criterion based on the value of the derivative cannot be applied because x^{**} is on the frontier of D. In this kind of situation, the solution is called a **corner solution**. The difficulty is that D is closed. On the contrary, if $D =]-3; 2[$ (D is open), f has no maximum but the minimum stays unchanged at $x^* = 0$.

These preliminary remarks show that solving an optimization problem

Download free eBooks at bookboon.com

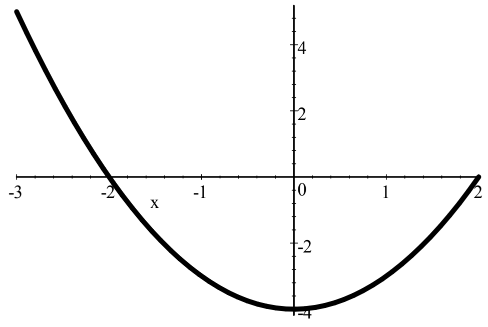

Figure 3.1: The function $f(x) = x^2 - 4$

using the successive derivatives works well if the domain is open. The solution can be more complicated to find when the domain is closed.

3.1.2 Regularity of the function to be optimized

The second feature playing a role in optimization programs is to know if the function to be optimized is sufficiently regular. Of course, if optimization criteria are based on derivatives the least we can ask is that these derivatives exist.

For example, consider the function defined on $\mathbb{R}$ by $f(x) = |x|$; f reaches its minimum for $x = 0$, but f is not differentiable at 0 (see figure 3.2). In fact, the derivative of f is nowhere equal to 0. The kind of irregularity observed at 0 is not that "wild", because f possesses at that point a right-derivative and a left-derivative. Nevertheless, no simple criterion can be found to solve the problem.

The example of $f(x) = |x|$ allows to understand why standard optimization conditions presented in the next sections require that the functions to be optimized are sufficiently regular. In financial applications, this regularity assumption is not too constraining or, more precisely, seems reasonable in a

Download free eBooks at bookboon.com

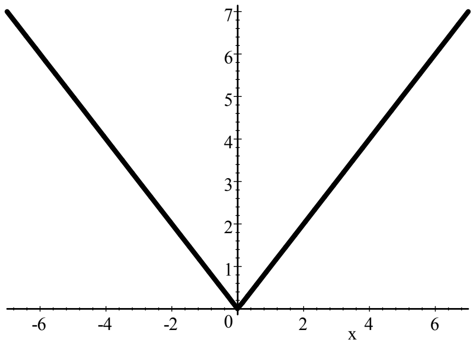

Figure 3.2: The function $f(x) = |x|$

number of circumstances.

3.1.3 Local and global optimum

Consider the function $f(x) = x\sin(x)$ depicted on figure 3.3; the graph is limited to the domain $D = [-7; +7]$. This function does not often show up in financial models but it is nevertheless interesting to understand the distinction between different types of optima.

First, f is regular and possesses all derivatives you might need. Second, it is clear that there are several points where the first derivative is equal to 0.

However, we immediately observe that the natural criterion of a null first derivative is not sufficient to distinguish maxima and minima. Looking at second-order derivatives allows to distinguish the two but only locally, that is on a short interval around the optimum under consideration.

The fact that derivatives provide local conditions to optimize functions is a serious problem in practical issues because all methods based on derivatives provide at best a local optimum (they are called *gradient methods*).

Finally, in this preliminary analysis, we have to mention that even the two

Download free eBooks at bookboon.com

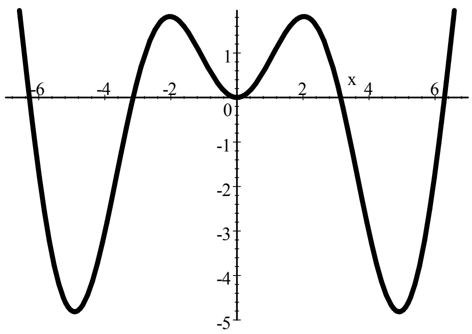

Figure 3.3: The function $f(x) = x\sin(x)$ on the domain $[-7; +7]$

first derivatives are often not sufficient to identify a maximum or a minimum.

Look at figure 3.4 that represents the function $f(x) = x^3$ over the domain $D = [-2; +2]$. The first two derivatives of f are equal to 0 at $x^* = 0$. However, the function has neither a minimum nor a maximum at x^*. One more time, a first derivative equal to 0 does not guarantee the existence of an optimum, without assuming something else.

Consider now the function $f(x) = x^4$ (figure 3.5); the function reaches its minimum at $x^* = 0$ with the two first derivatives equal to 0 at x^*.

In short, all these examples show that life may be complicated when it comes to optimizing. It is the reason why the different cases are examined in some details in the following sections.

Download free eBooks at bookboon.com

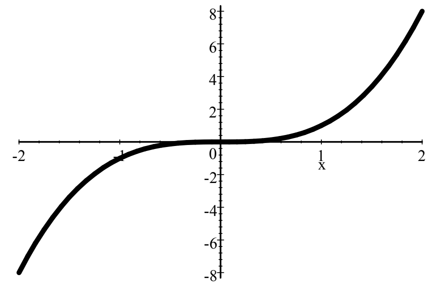

Figure 3.4: The function $f(x) = x^3$

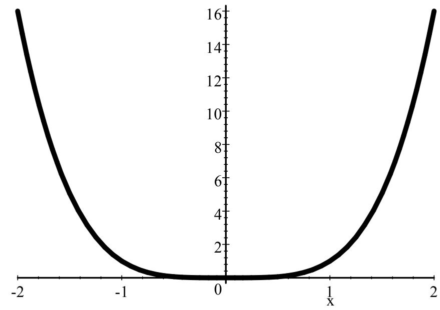

Figure 3.5: The function $f(x) = x^4$

3.2 Optimizing a single-variable function

We start by the most simple case: a function f depending on a single variable x. We are going to characterize minima and maxima of f, defined on a domain $D \subset \mathbb{R}$ and taking values in $\mathbb{R}$.

$$x \in D \to f(x) \in \mathbb{R} \tag{3.4}$$

The examples of the preceding section show that restrictions are necessary, either on D or on f, to obtain tractable optimality conditions. The first

Download free eBooks at bookboon.com

restriction, valid for the remainder of the section, is the following.

Assumption: D is an open subset of $\mathbb{R}$ and the functions considered in this section are twice continuously differentiable.

To avoid going back to part I of the book, we recall hereafter the definition of global and local optima.

Definition 124 *a) x_0 is a **local maximum** (minimum) of f if:*

$$\forall x \in]x_0 - \varepsilon; x_0 + \varepsilon, [\, f(x_0) \geq (\leq)\, f(x)$$

*b) x_0 is a **global maximum** (minimum) of f if there exists $\varepsilon > 0$ such that:*

$$\forall x \in D, f(x_0) \geq (\leq)\, f(x)$$

3.2.1 Necessary conditions of optimality

Proposition 125 *If x_0 is a local optimum of f then $f'(x_0) = 0$*

Keep in mind that this condition is necessary, not sufficient. You need to know that x_0 is an optimum to say that the first-derivative is equal to 0 at x_0. To emphazise the intuition that drives the result, consider the case of a local minimum. In a narrow interval including x_0, the function f is decreasing (increasing) on the left (right) of x_0 (otherwise x_0 would not be a minimum). Therefore, f' is negative on the left of x_0 and positive on the right. But we assumed that f is at least twice continuously differentiable; it means in particular that f' is continuous. A continuous function being negative (positive) on the left(right) of x_0 is equal to 0 at x_0.

Of course, a necessary condition is not very useful to solve empirical problems because in such problems we are looking for x_0; in most cases we do not look for properties of f at x_0 when we know that x_0 is an optimum.

It is the reason why sufficient conditions are more popular.

Download free eBooks at bookboon.com

3.2.2 Sufficient conditions of local optimality

Proposition 126 x_0 *is a local maximum (minimum) of* f *if:*

a) $f'(x_0) = 0$

b) $f"(x_0) < (>)0$

This proposition comes from the Taylor series expansions presented in part I of the book for single-variable functions and in the preceding chapter for functions depending on several variables. In fact, we can write:

$$f(x_0 + h) = f(x_0) + hf'(x_0) + \frac{h^2}{2} f"(x_0) + \varepsilon(h^2)$$

If $f'(x_0) = 0$ and $\varepsilon(h^2)$ is negligible with respect to h^2, the difference $f(x_0+h) - f(x_0)$ has the sign of $f"(x_0)$. If $f"(x_0) < 0$, then $f(x_0) > f(x_0+h)$ meaning that x_0 is a local maximum of f. Figure 3.1 is an illustration of the proposition. The minimum is obtained at $x_0 = 2$ and the derivative is increasing over an interval including x_0. Therefore, the derivative of f' is positive but this derivative is $f"$.

Remark 127 *Proposition 126 gives a sufficient condition but this condition is not necessary. The function* $f(x) = x^4$ *represented on figure 3.5 provides a good counter-example. In fact, there is a minimum at 0 but the two first derivatives are equal to 0. In general, for polynomials like* x^n*, a minimum exists if* n *is even and an inflection point appears for* n *odd. This remark justifies the general result hereafter.*

3.2.3 Necessary and sufficient optimality conditions

Proposition 128 x_0 *is a local maximum (minimum) of* f *if and only if*[2]*:*

a) $f'(x_0) = 0$

b) The order of the first non zero derivative at x_0 *is even and the corresponding derivative is negative (positive).*

[2] "if and only if" is often shortened in "iff".

Download free eBooks at bookboon.com

The optimality conditions presented so far are local optimality conditions. To obtain global conditions, we need to impose more assumptions on the behavior of f. The idea is that f should not be "authorized" to behave like $x\sin(x)$ with multiple changes in the sign of derivatives.

3.2.4 Global optimality conditions

Proposition 129 *If f is concave (convex) on the open convex domain D, x_0 is a global maximum (minimum) of f if $f'(x_0) = 0$.*

This result provides a very simple optimality condition only depending on the first derivative of f. Of course, the simplicity of the result comes from the concavity/convexity assumption which determines the sign of the second-order derivative. Knowing that many functions in finance or microeconomics problems satisfy this concavity/convexity assumption[3] is important. In such problems, checking if the first-order derivative is equal to 0 is enough to characterize a global optimum of f.

[3] The optimisation problems to solve are called "concave problems" in this case.

Download free eBooks at bookboon.com
Click on the ad to read more

Corollary 130 *If f is strictly concave (convex), the first-order condition provides the unique optimum.*

All the propositions of this section refer to single-variable functions. However, the geometric approach underlying the results is general. If x_0 is an optimum, the first-order condition says that the tangent to the curve representing f at x_0 is horizontal (its slope is 0). In the same spirit, the second-order condition is justified by the second-order Taylor series expansion which determines the sign of the changes in f around x_0. In the next section devoted to the optimization of functions depending on several variables, the tools are different, maybe a little bit more complex, but the logic and the geometry of the problem remain the same.

There is no difficulty to address a 30-variable problem when you know how to deal with a 29-variable problem. The "difficult" step is from single-variable functions to two-variable functions. It is the reason why we introduce an intermediate section devoted to the optimization of functions of two variables.

3.3 Optimizing a function of two variables

Devoting some place to optimizing functions of two variables is justified by the fact that such functions are represented by surfaces in three-dimensional spaces. It is then possible to draw their graphs, even if we are limited to two dimensions on the paper. With more than two variables, no graph can be drawn (as far as I know!) by standard means. Figure 3.6 is an example of the graph of a two-variable function.

The function f is defined by :

$$f(x_1, x_2) = \exp\left(-x_1^2 - x_2^2\right)$$

f has a maximum at $(0, 0)$ where it is worth 1, because $\exp(0) = 1$.

Download free eBooks at bookboon.com

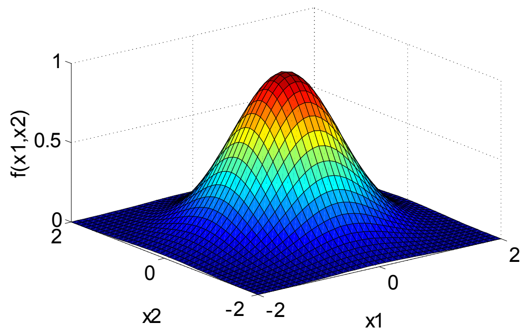

Figure 3.6: The function $f(x_1, x_2) = \exp(-x_1^2 - x_2^2)$

Suppose now that the second variable x_2 is constant, equal to 1. $f(x_1, x_2)$ becomes the single-variable function $g(x_1)$ defined by:

$$g(x_1) = f(x_1, 1) = \exp(-x_1^2 - 1)$$

g is represented on figure 3.7.

$$\exp(-x_1^2 - 1)$$

g reaches a maximum at $x_1^* = 0$ and its derivative equals 0 at x_1^*. In the same spirit we can define $h(x_2)$ by keeping x_1 constant. In such a case, h also has a maximum at $x_2 = 0$ with a null derivative at that point. But remember that keeping one variable constant is exactly what we did in the first part of the book to define partial derivatives of $f(x_1, x_2)$.

These remarks mean that partial derivatives are important in characterizing optima in multidimensional problems. The definitions of g and h and the properties of their derivatives show that two directions (along the x-axis and along the x_2-axis) should be considered when dealing with f. A maximum (x_1^*, x_2^*) of f should be a maximum for g when the value of x_2 is fixed to x_2^*

Download free eBooks at bookboon.com

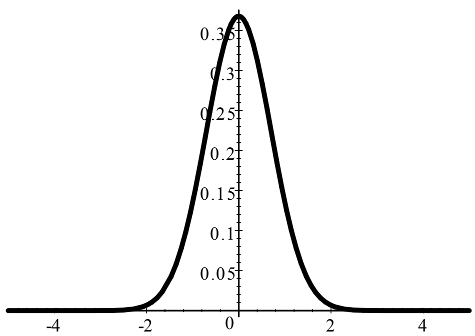

Figure 3.7: La fonction $f(x_1, 1) = \exp(-x_1^2)$

and a maximum for h when the value of x_1 is fixed to x_1^*. The geometric interpretation of this intuition is that the two-dimensional space tangent to the graph of f should be horizontal, that is parallel to the (x_1, x_2) plane. In fact, if it is not the case, we could find directions toward which f increases, a contradiction if (x_1^*, x_2^*) is a maximum.

The following subsections formalize the intuitions we just described. As in the preceding section we assume the following.

Assumption: D is an open subset of $\mathbb{R}^2$ and the functions considered in this section are twice continuously differentiable.

Continuous second-order partial derivatives ensure that the Hessian matrix is symmetric. For applications in finance and economics, it is not a restrictive assumption.

3.3.1 Local optimality conditions

Proposition 131 *If $x^* = (x_1^*, x_2^*)$ is a local optimum of f, then:*

$$\frac{\partial f}{\partial x_1}(x^*) = \frac{\partial f}{\partial x_2}(x^*) = 0$$

This proposition formalizes the intuition we just described by means of

Download free eBooks at bookboon.com

functions g and h. In the neighborhood of a maximum x^*, the values of f are lower than $f(x^*)$, especially toward the directions of x_1 and x_2 (that is if f is replaced by g or h). The conditions on partial derivatives say nothing else. These conditions can be shortened by writing $\nabla f(x^*) = 0$ where $\nabla f(x^*)$ is the gradient of f at x^*, (remember that the gradient is the vector of partial derivatives).

Of course, the gradient condition cannot be sufficient, simply because it does not allow to distinguish minima and maxima. Moreover, we already showed for single-variable functions that inflection points can exist. For functions depending on two variables, other more tricky situations can appear. Consider the function f defined by :

$$f(x) = x_1^2 - x_2^2$$

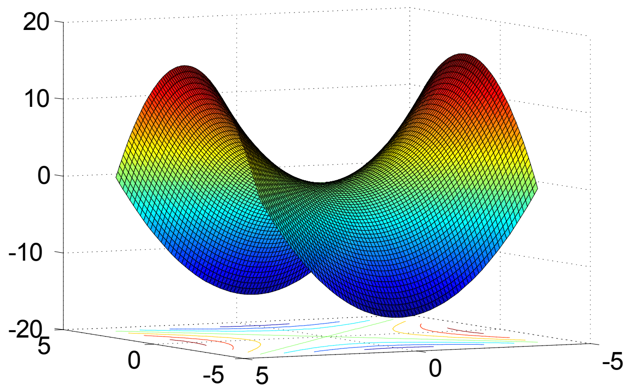

Figure 3.8: Example of a saddle point

The two first-order partial derivatives are equal to 0 at $x = (0; 0)$. In

Download free eBooks at bookboon.com

fact, these derivatives are equal to:

$$\frac{\partial f}{\partial x_1} = 2x_1 \text{ and } \frac{\partial f}{\partial x_2} = -2x_2$$

However, $x = (0; 0)$ is neither a maximum nor a minimum. The problem comes from the fact that, on one side, g (as a function of x_1 only) is convex and has a minimum at 0, but, on the other side, h (as a function of x_2 only) is concave and has a maximum at 0. This kind of situation is called a saddle point because, as you can see on figure 3.8, the graph of f in the neighborhood of $(0, 0)$ looks like a horse saddle.

This example shows that obtaining sufficient optimality conditions is going to require some precautions, even for local optima. In part I, we showed that there are n^2 second-order partial derivatives for a function depending on n variables. Therefore, we have 4 elements in the Hessian matrix for our functions depending on two variables. Eventually, the properties of the Hessian matrix are driving the sufficient conditions of optimality. They also allow to distinguish between optima and saddle points.

Download free eBooks at bookboon.com
Click on the ad to read more

Proposition 132 *x^* is a local maximum (minimum) of f if the following conditions are satisfied:*

1) $\nabla f(x^*) = 0$

2) $H_f(x^*)$ *is negative (positive) definite*

As for single-variable functions, the proof of this proposition is based on Taylor's formula. Let us denote $h' = (h_1, h_2)$; we can approximate $f(x^* + h)$ as follows:

$$f(x^* + h) = f(x^*) + h'\nabla f(x^*) + \frac{1}{2}h'H_f(x^*)h + \varepsilon(\|h\|^2)$$

$f(x^* + h) - f(x^*)$ and $h'H_f(x^*)h$ have the same sign when condition (1) is satisfied; if $H_f(x^*)$ is negative definite, $h'H_f(x^*)h < 0$, and then $f(x^*) > f(x^* + h)$. Symetrically, if $H_f(x^*)$ is positive definite, x^* is a local minimum.

Positive and negative definite matrices have been characterized in chapter 1. Using this characterization, proposition 132 can be rewritten as follows.

Corollary 133 *x^* is a local maximum (minimum) of f if:*

1) $\nabla f(x^*) = 0$

2) $\frac{\partial^2 f}{\partial x_1^2}(x^*) < (>)0$ *and* $Det(H_f(x^*)) > 0$

In fact, for a matrix to be negative definite, the signs of its principal minors must alternate, the first one being negative. For a matrix to be positive definite, all principal minors must be positive.

Looking more closely to the corollary can give the false idea that variable 1 is more important than variable 2. Of course, it is not the case because the determinant of $H_f(x^*)$ writes

$$Det(H_f(x^*)) = \frac{\partial^2 f}{\partial x_1^2}(x^*)\frac{\partial^2 f}{\partial x_2^2}(x^*) - \left[\frac{\partial^2 f}{\partial x_1 \partial x_2}(x^*)\right]^2$$

If this determinant is positive, the two second-order partial derivatives $\frac{\partial^2 f}{\partial x_1^2}(x^*)$ and $\frac{\partial^2 f}{\partial x_2^2}(x^*)$ have the same sign because the product of the two is positive.

Download free eBooks at bookboon.com

But the formulation of the corollary says nothing about what happens when the determinant is not strictly positive.

Considering the example presented at the beginning of the section ($f(x_1, x_2) = x_1^2 - x_2^2$) leads to:

$$H_f(x) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

and $Det(H_f(x)) = -4$. The signs of the principal minors alternate but the first one is positive and the determinant is negative. These features characterize a saddle point.

3.3.2 Global optimality conditions

The reasoning is exactly the same as the one we used for single-variable functions. To obtain global optimality conditions, we need to impose some restrictions (convexity or concavity) on the behavior of f.

We then obtain the following proposition.

Proposition 134 *If f, defined on a convex $D \subset \mathbb{R}^2$, is concave (convex), x^* is a global maximum(minimum) if $\nabla f(x^*) = 0$.*

This proposition is a direct generalization of proposition 129. The global optimum is obtained by means of a first-order condition because second-order conditions are automatically satisfied when f is concave (for a maximum) or convex (for a minimum). Corollary 130 can be rewritten for functions of two variables without changing a single word.

Corollary 135 *If f is strictly concave (convex), the first-order condition provides the unique maximum (minimum).*

Figure 3.9 represents the function $f(x) = x_1^2 + x_2^2$ which reaches its minimum at $x^* = (0,0)$. The principal minors of $H_f(x)$ are positive because the Hessian matrix is diagonal, each term of the diagonal being equal to 2.

Download free eBooks at bookboon.com

We observe on figure 3.9 that the null gradient at $(0,0)$ is equivalent to a horizontal tangent plane at that point.

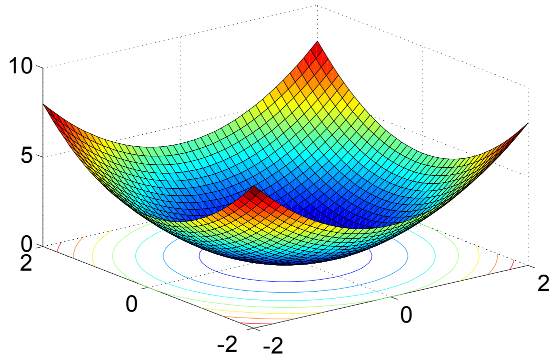

Figure 3.9: Horizontal tangent plane at the minimum of $f(x_1, x_2) = x_1^2 + x_2^2$

Many problems in finance assume the convexity (concavity) of the function to be minimized (maximized). Therefore, solutions often come by means of first-order conditions only, even if the concavity (convexity) of the function is not recalled systematically. For example, it is not always recalled that utility functions are assumed concave because it is a standard assumption in 99.9% of the models.

3.4 Functions of n variables

The general case of functions depending on n variables is not very different from the case $n = 2$ addressed in the preceding section....except that we cannot visualize the functions. The consequence of this proximity is that the statements of this section, especially the optimality conditions, are almost the same as the ones developed in the preceding section.

All the functions considered hereafter are defined on an open set $D \subset \mathbb{R}^n$. As usual, they are supposed twice continuously differentiable.

Download free eBooks at bookboon.com

3.4.1 Local optimality conditions

Proposition 136 *If x^* is a local optimum of f, then the gradient of f at $x^*$is 0.*

Remember that, in problems with two variables, this condition means that the tangent plane at x^* is horizontal, that is parallel to the plane x_1Ox_2. In $\mathbb{R}^n$ surfaces are called hypersurfaces and planes are hyperplanes. The meaning of "horizontal" is not intuitive in higher dimensions...but the idea is still the same. If f is reduced to a single-variable function by fixing, say, the values of the last $n-1$ variables, the partial derivative of f with respect to the first variable must be 0 if x^* is an optimum. If it was not the case, we could find a direction toward which the function f increases (for a maximum) or decreases (for a minimum). It would be a clear contradiction.

Proposition 137 *x^* is a local maximum (minimum) of f if the gradient of f is zero at x^* and the Hessian matrix is negative(positive) definite at x^*.*

Corollary 138 *1) x^* is a local maximum of f if its gradient is 0 at x^* and if the signs of the principal minors of $H_f(x^*)$ alternate, the first one being negative.*

2) x^* is a local minimum of f if its gradient is 0 at x^* and if the principal minors of $H_f(x^*)$ are positive.

We let the reader check that corollary 133 is a special case of the above corollary.

Example 139 *In this example, we are going to show how to build a term structure of (continuous) interest rates in a very simple case. We assume that three bonds are traded with respective maturities 1, 2 and 4 years. Table 3.1 summarizes the data.*

We assume a simple term structure of the following form:

$$r_t = a + bt^{0.8}$$

Download free eBooks at bookboon.com

Bond	Maturity	Coupon rate	Price(in %)
XXX	1	6%	101
YYY	2	5%	99.5
ZZZ	4	5.5%	100.5

Table 3.1: Bonds description

where t denotes the horizon under consideration, a and b are parameters to be estimated by minimizing the sum of the squares of the differences between observed prices and estimated prices. Note that if only the first two bonds are considered, a and b can be estimated without errors on prices. In fact, we should solve:

$$\begin{aligned} 101 &= 106\exp(-a-b) \\ 99.5 &= 5\exp(-a-b) + 105\exp(-a-b2^{0.8}) \end{aligned}$$

The first equality is justified because the first bond pays a unique cash-flow of 106 in 1 year (the coupon rate is 6%). The second equality states the equality of the price and of the sum of the discounted cash-flows for the second bond. Solving these simple equations leads to:

$$\begin{aligned} a &= -2.5287 \times 10^{-2} \\ b &= .0736 \end{aligned}$$

Applying this estimation to the last bond gives a theoretical price of:

$$\sum_{t=1}^{4} 5.5*\exp(2.5287\times10^{-2} - .0736t^{0.8}) + 105.5\exp(2.5287\times10^{-2} - .0736\times4^{0.8}) = $$

But the market price is 100.5. There is a large difference, meaning that a perfect match of the three prices is impossible.

Download free eBooks at bookboon.com

a and b must be estimated by minimizing the following function:

$$f(a,b) = \sum_{i=1}^{3} (\pi_i - \widehat{\pi}_i)^2$$

where π_i is the market price and $\widehat{\pi}_i$ is the theoretical price.

Of course, in practice the problem is not solved manually but using a computer program or, at least, a spreadsheet. For example, the Excel Solver can easily solve this problem.

Download free eBooks at bookboon.com
Click on the ad to read more

3.4.2 Global optimality conditions

We can repeat word by word what we said for functions of two variables; we just need to adapt the dimensions. To obtain global optimality conditions, we impose convexity or concavity of f.

We then obtain the following proposition.

Proposition 140 *If f, defined on a convex subset $D \subset \mathbb{R}^n$, is concave (convex), x^* is a global maximum(minimum) if $\nabla f(x^*) = 0$.*

The global optimum is obtained by means of a first-order condition because second-order conditions are automatically satisfied (f concave for a maximum and f convex for a minimum).

Download free eBooks at bookboon.com

Corollary 141 *If f is strictly concave (convex), the first-order condition provides the unique optimum.*

This corollary is exactly the same as corollary 135. The reader understands now why it was useful to devote some place to functions depending on two variables.

Download free eBooks at bookboon.com